ON THE CAYLEY DEGREE OF AN ALGEBRAIC GROUP

NICOLE LEMIRE, VLADIMIR L. POPOV, AND ZINOVY REICHSTEIN

ABSTRACT. A connected linear algebraic group G is called a *Cayley group* if the Lie algebra of G endowed with the adjoint G-action and the group variety of G endowed with the conjugation G-action are birationally G-isomorphic. In particular, the classical Cayley map

$$X \mapsto (I_n - X)(I_n + X)^{-1}$$

between the special orthogonal group $\mathbf{SO}_n$ and its Lie algebra $\mathfrak{so}_n$, shows that $\mathbf{SO}_n$ is a Cayley group. In an earlier paper we classified the simple Cayley groups defined over an algebraically closed field of characteristic zero. Here we consider a new numerical invariant of G, *the Cayley degree*, which "measures" how far G is from being Cayley, and prove upper bounds on Cayley degrees of some groups.

1. INTRODUCTION

Let G be a connected linear algebraic group and let $\mathfrak{g}$ be its Lie algebra. We say that G is a *Cayley group* if there is a birational isomorphism

$$\varphi\colon G \dashrightarrow \mathfrak{g} \tag{1}$$

which is equivariant with respect to the conjugation action of G on itself and the adjoint action of G on $\mathfrak{g}$; see [LPR, Definition 1.5]. In particular, the classical Cayley map [C]

$$X \mapsto (I_n - X)(I_n + X)^{-1} \tag{2}$$

between the special orthogonal group $\mathbf{SO}_n$ and its Lie algebra $\mathfrak{so}_n$ shows that $\mathbf{SO}_n$ is a Cayley group. (The same formula shows that $\mathbf{Sp}_{2n}$ is Cayley as well.) In the sequel we will always assume that the base field k is algebraically closed and of characteristic zero. (Problem 1 below is of interest for arbitrary k but the partial answers we would like to discuss here require this assumption.)

In 1975 D. Luna [L_1], [L_3] asked the second-named author a question that, in the above terminology, can be restated as follows: For what n is the group $\mathbf{SL}_n$ Cayley? In [LPR] we showed that $\mathbf{SL}_n$ is Cayley if and only if $n \leqslant 3$ and, more generally, proved the following classification theorem.

Theorem 1. ([LPR, Theorem 3.31(a)]) *A connected simple algebraic group G is Cayley if and only G is isomorphic to one of the following groups:* $\mathbf{SL}_2$, $\mathbf{SL}_3$, $\mathbf{SO}_n$ $(n \neq 2, 4)$, $\mathbf{Sp}_{2n}$, $\mathbf{PGL}_n$ $(n \geqslant 1)$.

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Note that $\mathbf{SO}_n$ is a Cayley group for every $n \geqslant 1$; we have excluded $\mathbf{SO}_2$ and $\mathbf{SO}_4$ from the above list because these groups are not simple.

A *generalized Cayley map* of G is a rational G-equivariant map $\varphi\colon G \dashrightarrow \mathfrak{g}$, as in (1), except that instead of requiring it to be a birational isomorphism, we only require it to be dominant, see [LPR, Definition 10.9]. Every generalized Cayley map of G has finite degree,

$$\deg \varphi = [k(G) : \varphi^*(k(\mathfrak{g}))] < \infty,$$

where, as usual, $k(X)$ and $k[X]$ denote respectively the field of rational and the algebra of regular functions on an irreducible algebraic variety X). A generalized Cayley map (1) exists for every linear algebraic group G; see [LPR, Proposition 10.5]. Hence the following natural number is well defined.

Definition 1. The *Cayley degree* $\mathrm{Cay}(G)$ of G is the minimal value of $\deg \varphi$, as φ ranges over all generalized Cayley maps of G.

Note that, by definition, G is a Cayley group if and only if $\mathrm{Cay}(G) = 1$. Therefore Theorem 1 may be viewed as a first step toward a solution of the following more general problem.

Problem 1. *Find the Cayley degrees of connected simple algebraic groups.*

We do not have any general methods for proving lower bounds on the Cayley degree, beyond those provided by Theorem 1; in particular, we do not have an example of a linear algebraic group G with $\mathrm{Cay}(G) > 2$. Thus in this note we will primarily concentrate on upper bounds. Our main results are Theorems 2 and 3 below.

Theorem 2. *If $n \geqslant 3$, then $\mathrm{Cay}(\mathbf{SL}_n) \leqslant n-2$.*

Our proof of Theorem 2 is self-contained. For $n = 3$ this argument gives a new proof of the fact that $\mathrm{Cay}(\mathbf{SL}_3) = 1$ (i.e., $\mathbf{SL}_3$ is a Cayley group), which is simpler than either of the two proofs in [LPR]. For $n = 4$, Theorem 2 implies that $\mathrm{Cay}(\mathbf{SL}_4) = 2$; see Example 4.

To motivate our second main result, we note that the exceptional group $\mathbf{G}_2$ plays a special role in this theory. While $\mathbf{G}_2$ is not a Cayley group, it is close to being one, in the sense that $\mathbf{G}_2 \times \mathbf{G}_m^2$ is Cayley; see [LPR, Theorem 1.31(b)]. In fact, $\mathbf{G}_2$ is the unique simple group G which is stably Cayley but is not Cayley; see [LPR, Theorems 1.29 and 1.31]. (Recall that G is called *stably Cayley* if $G \times \mathbf{G}_m^r$ is Cayley for some $r \geqslant 1$.) Theorem 3 below shows that $\mathbf{G}_2$ is also close to being Cayley in the sense of having a small Cayley degree.

Theorem 3. $\mathrm{Cay}(\mathbf{G}_2) = 2$.

The rest of this note is structured as follows. In Section 2 we determine the Cayley degrees of Spin groups and some groups of type A. In Section 3 we prove a lemma that reduces the computation of the Cayley degree of a reductive group G to a question about finite group actions. This lemma is then used as a starting point for the proofs of Theorems 2 and 3 in Sections 4 and 5 respectively. In Section 6 we give a representation theoretic interpretation of the Cayley degree.

2. First examples

Lemma 1. (a) *Let $\pi\colon G \to H$ be an isogeny between connected linear algebraic groups and let d be the order of its kernel.*

(a_1) *Then*

$$\mathrm{Cay}(G) \leqslant d \cdot \mathrm{Cay}(H)\,.$$

(a_2) *If G is not Cayley but H is Cayley, and $d = 2$, then* $\mathrm{Cay}(G) = 2$.

(b) *Let φ_i be a generalized Cayley map of a connected linear algebraic group G_i, where $i = 1, \ldots, n$. Then $\varphi_1 \times \ldots \times \varphi_n$ is a generalized Cayley map of $G_1 \times \ldots \times G_n$, and*

$$\deg\,(\varphi_1 \times \ldots \times \varphi_n) = \deg\,\varphi_1 \ldots \deg\,\varphi_n.$$

Proof. (a_1) The groups G and H have the same Lie algebra $\mathfrak{g}$. Let $\varphi\colon H \dashrightarrow \mathfrak{g}$ be a generalized Cayley map of H. Since $\operatorname{Ker}\pi$ is a finite central subgroup of G and $\deg\pi = d$, the composition $\varphi\circ\pi\colon G \dashrightarrow \mathfrak{g}$ is a generalized Cayley map of G. Its degree is $d\cdot\deg\,\varphi$, and part (a_1) follows.

(a_2) Since G is not Cayley, we have $\mathrm{Cay}(G) \geqslant 2$. The opposite inequality follows from part (a_1).

Part (b) follows from the interpretation of degree of a rational map as the number of points in a general fiber. ☐

From (b) and Definition 1 we obtain the following upper bound.

Corollary 1. $\mathrm{Cay}(G_1 \times \ldots \times G_n) \leqslant \mathrm{Cay}(G_1)\cdot \ldots \mathrm{Cay}(G_n)$.

The following example shows that, in general, equality does not hold.

Example 1. Since $\mathrm{Cay}(\mathbf{G}_2) \geq 2$ by Theorem 1, but $\mathrm{Cay}(\mathbf{G}_2 \times \mathbf{G}_m^2) = 1$ (see [LPR, Theorem 1.31]), we see that

$$\mathrm{Cay}(\mathbf{G}_2 \times \mathbf{G}_m^2) < \mathrm{Cay}(\mathbf{G}_2)\cdot \mathrm{Cay}(\mathbf{G}_m^2)\,.$$

(In fact, the right hand side of this inequality is equal to 2, because $\mathrm{Cay}(\mathbf{G}_2) = 2$ by Theorem 2 and $\mathrm{Cay}(\mathbb{G}_m^2) = 1$; see [LPR, Example 1.21].)

Example 2. (see [LPR, p. 962]) The groups

$$\mathbf{Spin}_2 \simeq \mathbb{G}_m\,, \quad \mathbf{Spin}_3 \simeq \mathbf{SL}_2\,, \quad \mathbf{Spin}_4 \simeq \mathbf{SL}_2 \times \mathbf{SL}_2\,, \quad \mathbf{Spin}_5 \simeq \mathbf{Sp}_4$$

are easily seen to be Cayley. On the other hand, $\mathbf{Spin}_n$ is not Cayley if $n \geqslant 6$. Since $\mathbf{SO}_n$ is Cayley for every n, applying Lemma 1(b) to the natural 2-sheeted isogeny $\mathbf{Spin}_n \to \mathbf{SO}_n$ (where $n \geqslant 6$), we obtain

$$\mathrm{Cay}(\mathbf{Spin}_n) = \begin{cases} 2 & \text{for } n \geqslant 6, \\ 1 & \text{for } n \leqslant 5. \end{cases} \tag{3}$$

Example 3. Since $\mathbf{PGL}_n$ is a Cayley group for every $n \geqslant 1$, Lemma 1, applied to the natural isogeny $\mathbf{SL}_n/\boldsymbol{\mu}_d =: G \to H := \mathbf{PGL}_n$ yields

$$\mathrm{Cay}(\mathbf{SL}_n/\boldsymbol{\mu}_d) \leqslant n/d\,. \tag{4}$$

In particular,

$$\mathrm{Cay}(\mathbf{SL}_{2d}/\boldsymbol{\mu}_d) = \begin{cases} 2 & \text{for } d \geqslant 3, \\ 1 & \text{for } d \leqslant 2. \end{cases}$$

Note also that setting $d = 1$ in (4) yields $\mathrm{Cay}(\mathbf{SL}_n) \leqslant n$. Theorem 2 strengthens this bound.

3. The maximal torus

In this section we reduce the problem of finding $\mathrm{Cay}(G)$ for a connected reductive group G, to a question about finite group actions.

Lemma 2. *Let G be a connected linear algebraic group, let T be its maximal torus, let C and N be the centralizer and normalizer of T in G respectively, and let $W := N/C$ be the Weyl group. Denote the Lie algebras of G, T, and C by $\mathfrak{g}$, $\mathfrak{t}$, and $\mathfrak{c}$, respectively.*

(a) *Then*
$$\mathrm{Cay}(G) = \min_{\psi} \deg \psi, \tag{5}$$
where ψ ranges over all dominant rational N-equivariant maps $C \dashrightarrow \mathfrak{c}$.

(b) *Moreover, if G is reductive, then* (5) *holds, where ψ ranges over all W-equivariant dominant rational maps $T \dashrightarrow \mathfrak{t}$.*

Proof. Recall that $G \simeq G \times^N C$ and $\mathfrak{g} \simeq G \times^N \mathfrak{c}$, where $\simeq$ stands for a birational isomorphism of G-varieties. Moreover, if $\varphi\colon G \times^N C \dashrightarrow G \times^N \mathfrak{c}$ is a dominant rational G-map, then $\psi := \varphi|_C : C \dashrightarrow \mathfrak{c}$ is a dominant rational N-map and $\varphi^{-1}(x) = \psi^{-1}(x)$ for a general point $x \in \mathfrak{c}$; see [LPR, Lemma 2.17]. Hence
$$\deg \varphi = |\varphi^{-1}(x)| = |\psi^{-1}(x)| = \deg \psi. \tag{6}$$
Thus we have a degree preserving bijection between generalized Cayley maps of G and dominant rational N-equivariant maps $C \dashrightarrow \mathfrak{c}$. This immediately implies (a). If G is reductive, then $C = T$, $\mathfrak{c} = \mathfrak{t}$, and the N-actions on C and $\mathfrak{c}$ descend to the W-actions (since T, being commutative, acts trivially). Hence part (b) follows from part (a). □

Corollary 2. *Let φ be a generalized Cayley map of a connected reductive group G. Then* $\deg\, \varphi = [k(G)^G : \varphi^*(k(\mathfrak{g})^G)]$.

Proof. We will continue to use the notations of Lemma 2 and set $\psi := \varphi|_T$. Since W is a finite group acting on T and $\mathfrak{t}$ faithfully, we have $[k(T) : k(T)^W] = |W|$ and $[k(\mathfrak{t}) : k(\mathfrak{t})^W] = |W|$. From this we deduce that $\deg\, \psi := [k(T) : \psi^*(k(\mathfrak{t}))] = [k(T)^W : \psi^*(k(\mathfrak{t})^W)]$. Since we have $[k(T)^W : \psi^*(k(\mathfrak{t})^W)] = [k(G)^G : \varphi^*(k(\mathfrak{g})^G)]$, see [P, Theorem (1.7.5)], [LPR, (3.4)], the claim now follows from (6). □

Remark 1. If φ is a morphism, Corollary 2 can be deduced from [L_3, Lemme Fondamental]. For certain particular morphisms φ, a proof can be found in [KM, Corollary (3.3)].

4. Proof of Theorem 2

By Lemma 2 it suffices to construct a dominant rational $W = \mathrm{S}_n$-equivariant map between the maximal torus T in $\mathbf{SL}_n$ and its Lie algebra $\mathfrak{t}$.

To keep the notation clear in the construction to follow, we will work with two copies of the affine space $\mathbb{A}^n$, with the same natural (permutation) action of S_n. We will denote one by $\mathbb{A}^n_x$ and the other by $\mathbb{A}^n_y$ and use the variables $x_1, \dots, x_n$ and, respectively, $y_1, \dots, y_n$ as standard coordinate functions on $\mathbb{A}^n_x$ and $\mathbb{A}^n_y$. We will now embed $\mathfrak{t}$ and, respectively, T into $\mathbb{A}^n_x$ and $\mathbb{A}^n_y$ as the following S_n-invariant subvarieties:
$$\mathfrak{t} = \{(a_1, \dots, a_n) \in \mathbb{A}^n_x \mid a_1 + \dots + a_n = 0\},$$
$$T = \{(b_1, \dots, b_n) \in \mathbb{A}^n_y \mid b_1 \dots b_n = 1\}.$$

Consider the mutually inverse S_n-equivariant rational maps $\varphi\colon \mathbb{A}^n_x \to \mathbb{A}^n_y$ and $\psi\colon \mathbb{A}^n_y \to \mathbb{A}^n_x$ given by

$$\varphi := \left(\frac{x_1+1}{x_1}, \dots, \frac{x_n+1}{x_n}\right) \quad \text{and} \quad \psi := \left(\frac{1}{y_1-1}, \dots, \frac{1}{y_n-1}\right).$$

These maps give rise to a (biregular) isomorphism between the open subsets

$$U_x := \{(a_1, \dots, a_n) \in \mathbb{A}^n_x \mid a_1 \dots a_n \neq 0\}$$

and

$$U_y := \{(b_1, \dots, b_n) \in \mathbb{A}^n_y \mid (b_1-1) \dots (b_n-1) \neq 0\}$$

in $\mathbb{A}^n_x$ and $\mathbb{A}^n_y$ respectively. Substituting $y_i = \dfrac{x_i+1}{x_i}$ into the equation $y_1 \dots y_n - 1 = 0$ of T, we see that $\psi(T \cap U_y) = X \cap U_x$, where X is the hypersurface in $\mathbb{A}^n_x$ cut out by the equation

$$f(x_1, \dots, x_n) := (x_1+1) \dots (x_n+1) - x_1 \dots x_n = 0.$$

Since $X \cap U_x$ is isomorphic to $T \cap U_y$ (which is irreducible) and X does not contain any of the n components $\{x_i = 0\}$ of the complement of U_x, we conclude that X is irreducible S_n-invariant hypersurface in $\mathbb{A}^n_x$. Hence f is a power of an irreducible polynomial. Since $\deg f(1, \dots, 1, x_i, 1, \dots, 1) = 1$ for every i, we conclude that in fact f is irreducible. As $\deg f = n-1$, this implies that X is a hypersurface of degree $n-1$. By our construction X is birationally isomorphic to T (via φ), as an S_n-variety.

Let π be the projection $X \dashrightarrow \mathfrak{t}$ from a point $\boldsymbol{a} = (a, \dots, a) \in \mathbb{A}^n_x$. That is, for any point $\boldsymbol{b} \in X$, $\boldsymbol{b} \neq \boldsymbol{a}$, the point $\pi(\boldsymbol{b})$ is the intersection point of the line passing through $\boldsymbol{a}$ and $\boldsymbol{b}$ with the hyperplane $\mathfrak{t} \subset \mathbb{A}^n_x$. Moreover, we choose $\boldsymbol{a}$ so that it lies on X. Note that this automatically means that it does not lie in $\mathfrak{t}$. Indeed, since zero does not satisfy the equation

$$f(a, \dots, a) = (1+a)^n - a^n = 0,$$

if $\boldsymbol{a} \in X$, then $\boldsymbol{a}$ cannot lie in $\mathfrak{t}$. Since our base field k is algebraically closed and of characteristic zero, such an a exists for every $n \geqslant 2$. Note that π is well-defined, unless X is a hyperplane parallel to $\mathfrak{t}$. Since $\deg X = n-1$, it is not a hyperplane for every $n \geqslant 3$. Thus π is well-defined for every $n \geqslant 3$. Note also that since $\boldsymbol{a}$ is fixed by S_n, the map π is S_n-equivariant.

We claim that $\pi\colon X \dashrightarrow \mathfrak{t}$ is dominant. Since π is a projection map from a point on a hypersurface X, and $\deg X = n-1$, this claim implies that $\deg \pi = n-2$. Composing π with a birational isomorphism $\psi\colon T \dashrightarrow X$, we obtain an S_n-equivariant dominant rational map $T \dashrightarrow \mathfrak{t}$ of degree $n-2$, and Theorem 2 is proved.

It remains to show that π is dominant. Assume the contrary. Let X_0 be the closure of the image of π in $\mathfrak{t}$. Then X is the cone over X_0 centered at $\mathbf{a}$. Since, as we remarked above, X is not a hyperplane (we are assuming throughout that $n \geqslant 3$), X has to be singular at $\boldsymbol{a}$. Consequently, a satisfies the system of equations

$$\begin{cases} f(\boldsymbol{a}) = (1+a)^n - a^n = 0\,, \\ \dfrac{\partial f}{\partial x_1}(\boldsymbol{a}) = (1+a)^{n-1} - a^{n-1} = 0\,. \end{cases}$$

But this system has no solutions, a contradiction. Theorem 2 is now proved. □

Example 4. By Theorem 2, $\mathrm{Cay}(\mathbf{SL}_4) \leqslant 2$. Equivalently, $\mathrm{Cay}(\mathbf{SL}_4) = 2$; indeed, we know that $\mathrm{Cay}(\mathbf{SL}_4) \neq 1$, i.e., $\mathbf{SL}_4$ is not a Cayley group by Theorem 1.

Since $\mathbf{SL}_4/\boldsymbol{\mu}_2 \simeq \mathbf{SO}_4$ is Cayley, the equality $\mathrm{Cay}(\mathbf{SL}_4) = 2$ can also be obtained by applying Lemma 1(b) to the isogeny $\mathbf{SL}_4 \to \mathbf{SL}_4/\boldsymbol{\mu}_2$. Alternatively, since $\mathbf{SL}_4 \simeq \mathbf{Spin}_6$, the equality $\mathrm{Cay}(\mathbf{SL}_4) = 2$ is a special case of (3).

5. Proof of Theorem 3

First recall that $\mathbf{G}_2$ is not Cayley (see Theorem 1) and hence $\mathrm{Cay}(\mathbf{G}_2) \geqslant 2$. Thus we only need to prove the opposite inequality. By Lemma 2 it suffices to construct a W-equivariant dominant rational map $T \dashrightarrow \mathfrak{t}$ of degree 2, where T is a maximal torus of $\mathbf{G}_2$, $\mathfrak{t}$ is the Lie algebra of T, and W is the Weyl group.

Recall that W is isomorphic to $\mathrm{S}_3 \times \mathbb{Z}/2\mathbb{Z}$. Once again, we consider two copies of the 3-dimensional affine space, $\mathbb{A}^3_x$ and $\mathbb{A}^3_y$, with the following W-actions. The symmetric group S_3 acts on both copies in the natural way (by permuting the coordinates). The nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{A}^3_x$ by

$$(a_1, a_2, a_3) \mapsto (-a_1, -a_2, -a_3),$$

and on $\mathbb{A}^3_y$ by

$$(b_1, b_2, b_3) \mapsto \left(\frac{1}{b_1}, \frac{1}{b_2}, \frac{1}{b_3}\right).$$

We may (and shall) embed $\mathfrak{t}$ and T into $\mathbb{A}^3_x$ and $\mathbb{A}^3_y$, respectively, as the following W-invariant subvarieties:

$$\mathfrak{t} = \{(a_1, a_2, a_3) \in \mathbb{A}^3_x \mid a_1 + a_2 + a_3 = 0\},$$
$$T = \{(b_1, b_2, b_3) \in \mathbb{A}^3_y \mid b_1 b_2 b_3 = 1\}.$$

We now consider the mutually inverse W-equivariant rational maps $\varphi\colon \mathbb{A}^3_x \to \mathbb{A}^3_y$ and $\psi\colon \mathbb{A}^3_y \to \mathbb{A}^3_x$ given by

$$\varphi := \left(\frac{x_1 - 1}{x_1 + 1}, \frac{x_2 - 1}{x_2 + 1}, \frac{x_3 - 1}{x_3 + 1}\right) \quad \text{and} \quad \psi := \left(-\frac{y_1 + 1}{y_1 - 1}, -\frac{y_2 + 1}{y_2 - 1}, -\frac{y_3 + 1}{y_3 - 1}\right).$$

These maps give rise to a W-equivariant isomorphism between the open subsets

$$U_x := \{(a_1, a_2, a_3) \in \mathbb{A}^3_x \mid (a_1 + 1)(a_2 + 1)(a_3 + 1) \neq 0\}$$

and

$$U_y := \{(b_1, b_2, b_3) \in \mathbb{A}^3_y \mid (b_1 - 1)(b_2 - 1)(b_3 - 1) \neq 0\}$$

in $\mathbb{A}^3_x$ and $\mathbb{A}^3_y$, respectively. Substituting $y_i = \dfrac{x_i - 1}{x_i + 1}$ into the equation $y_1 y_2 y_3 = 1$ of T, we see that $\psi(T \cap U_y) = X \cap U_x$, where X is the W-invariant quadric surface in $\mathbb{A}^3_x$ defined by the equation

$$x_1 x_2 + x_2 x_3 + x_1 x_3 + 1 = 0.$$

Composing the W-equivariant birational isomorphism $\psi\colon T \dashrightarrow \mathbb{A}^3_x$ with the W-invariant linear projection $\alpha\colon X \to \mathfrak{t}$ given by

$$\alpha := \left(x_1 - \frac{x_1 + x_2 + x_3}{3}, x_2 - \frac{x_1 + x_2 + x_3}{3}, x_3 - \frac{x_1 + x_2 + x_3}{3}\right),$$

we obtain a desired W-equivariant rational map $\alpha \circ \psi\colon T \dashrightarrow \mathfrak{t}$ of degree 2. $\square$

Remark 2. The proofs of Theorems 2 and 3 proceed along similar lines: we begin by defining a birational isomorphism ψ between T and a hypersurface X, then project X onto $\mathfrak{t}$. Note, however, that the projections π (in the proof of Theorem 2) and α (in the proof of Theorem 3) are different in the following sense: π is a projection from a point on X, and α is a linear projection (α may also be viewed as a projection from a point at infinity, which does not lie on X). Note that α cannot be replaced by a projection from a point of X, since X has no W-equivariant points (and also because otherwise α would have degree 1 and our argument would show that $\mathbf{G}_2$ is a Cayley group, which we know to be false).

Remark 3. The formula for φ is somewhat similar to the formula for the classical Cayley map (2). Note, however, that we cannot replace $\dfrac{x_1-1}{x_1+1}, \dfrac{x_2-1}{x_2+1}$, etc. by $\dfrac{1-x_1}{x_1+1}, \dfrac{1-x_2}{x_2+1}$, etc. in the definition of φ. If we do this, then, setting $\psi = \varphi^{-1}$, we see that the image of T under ψ becomes the cubic $x_1x_2x_3 + x_1 + x_2 + x_3 = 0$, rather than the quadric $x_1x_2 + x_2x_3 + x_1x_3 + 1 = 0$, and the above argument gives a generalized Cayley map of degree 3, rather than 2.

6. A representation theoretic approach

In conclusion we outline a representation theoretic approach to determining the Cayley degree of an algebraic group.

Let X be an irreducible algebraic variety endowed with an action of an algebraic group H, and let V be a vector space over k of dimension $\dim X$ endowed with a linear action of H. Then rational dominant H-maps $X \dashrightarrow V$ are described as follows. Let M be a submodule of the H-module $k(X)$ such that

(i) M is isomorphic to the H-module V^*,
(ii) $k(X)$ is algebraic over the subfield $k(M)$ generated by M over k.

By (ii), $k(M)/k$ is a purely transcendental extension of degree $\dim X$. Since $k(V)$ is generated over k by V^*, any isomorphism of H-modules $V^* \to M$ can be uniquely extended up to an H-equivariant embedding $\iota\colon k(V) \hookrightarrow k(X)$ whose image is $k(M)$. This embedding determines a rational dominant H-map $\psi\colon X \dashrightarrow V$ such that $\psi^* = \iota$. We have

$$\deg\,\psi = [k(X) : k(M)]. \tag{7}$$

Any dominant rational H-map $X \dashrightarrow V$ is obtained in this way.

Now suppose G is a connected reductive linear algebraic group, $X = T$ is a maximal torus, $V = \mathfrak{t}$ is the Lie algebra of T and $H = W = N_G(T)/T$ is the Weyl group. In view of Lemma 2(b) the above approach relates generalized Cayley maps of G to the W-module structure of $k(T)$. This connection may be used to prove upper bounds on $\mathrm{Cay}(G)$.

Example 5. Let $G = \mathbf{G}_2$. Use the notation of Section 5. Let t_i be the restriction of y_i to T. Then $t_1t_2t_3 = 1$ and $k(T) = k(t_1, t_2)$. Put

$$z_i := t_i - t_i^{-1}. \tag{8}$$

From the description of the W-actions on T and $\mathfrak{t}$ given in Section 5 it follows that

$$M := \{\alpha_1z_1 + \alpha_2z_2 + \alpha_3z_3 \mid \alpha_1 + \alpha_2 + \alpha_3 = 0,\ \ \alpha_i \in k\} \tag{9}$$

is a submodule of the W-module $k(T)$ that is isomorphic to the W-module $\mathfrak{t}^*$. Let

$$s_1 := z_1 - z_2,\ s_2 := z_1 - z_3 \tag{10}$$

Then s_1, s_2 is a basis of M, so $k(M) = k(s_1, s_2)$. We have $k(t_1, s_1, s_2) = k(T)$ because $t_2 = (t_1^2 - 1)(t_1^2 s_1 + t_1 s_2 - t_1^3 - t_1^2 + t_1 + 1)^{-1}$. It follows from (8), (10) that

$$\begin{cases} -t_2 + t_2^{-1} = s_1 - t_1 + t_1^{-1}, \\ t_1 t_2 - t_1^{-1} t_2^{-1} = s_2 - t_1 + t_1^{-1}. \end{cases} \tag{11}$$

Eliminating t_2 and t_2^{-1} from (11), we obtain the following equation:

$$\begin{aligned} t_1^6 - (s_1 + s_2)t_1^5 + (s_1 s_2 - 2s_1 - 2s_2 - 1)t_1^4 + (s_1^2 + s_2^2 - 5)t_1^3 \\ + (s_1 s_2 + 2s_1 + 2s_2 + 1)t_1^2 + (s_1 + s_2 + 1)t_1 + 1 = 0. \end{aligned}$$

Thus for the conjugating and adjoint actions of $H := W$ respectively on $X := T$ and $V := \mathfrak{t}$, and for M defined by (9), the above conditions (i), (ii) hold and $[k(T) : k(M)] \leqslant 6$. Hence by (7), (6), and Lemma 2, there exists a generalized Cayley map of G of degree $[k(T) : k(M)]$. In particular, this implies that $\mathrm{Cay}(\mathbb{G}_2) \leqslant 6$ (of course, by Theorem 3, we know that in fact $\mathrm{Cay}(\mathbb{G}_2) = 2$). $\square$

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Department of Mathematics, University of Western Ontario, London, Ontario N6A 5B7, Canada

E-mail address: nlemire@uwo.ca

Steklov Mathematical Institute, Russian Academy of Sciences, Gubkina 8, Moscow 119991, Russia

E-mail address: popovvl@orc.ru

Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada

E-mail address: reichste@math.ubc.ca

URL: www.math.ubc.ca/~reichst